

**OPERATOR SEMIGROUPS ACTING ON A  $\Gamma$ -SEMIGROUPS**

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**Abstract**

The concept of  $\Gamma$ -semigroup is a generalization of semigroups. In this paper, we briefly introduce the action of left (right) operator semigroups on a  $\Gamma$ -semigroup and deduce in particular that there exists an inclusion preserving bijection between the set of all right ideals of  $S$  and the set all right ideals of  $L \times S$ .

**Keywords:**  $\Gamma$ -semigroup; Operator semigroup.

# 1 Introduction

The notion of  $\Gamma$  in algebra was first introduced by Nobusawa [8] as a generalization of ring in the form of  $\Gamma$ -ring. Let  $M$  and  $\Gamma$  be additive groups such that for all  $a, b, c \in M$  and  $\gamma, \beta, \alpha \in \Gamma$ , we have  $a\gamma b \in M$  and  $\gamma a\beta \in \Gamma$  for every  $a, b, \gamma$  and  $\beta$ , then  $M$  is called a  $\Gamma$ -ring if the following conditions are satisfied:

- (i)  $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b$ ,  
 $a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$ ,  
 $a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2$ ,
- (ii)  $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$ ,
- (iii) if  $a\gamma b = 0$  for any  $a, b \in M$ , then  $\gamma = 0$ .

The structure of  $\Gamma$ -rings as initiated by [8] was studied by Barnes [1], Luh [7], Ravishankar and Shukla [9], Buys and Groenewald [3], Booth [2] and Kyuno [6]. Motivated by this generalization of a ring, Sen [13] defined the concept of  $\Gamma$ -semigroup. Later, Sen and Saha [14] redefined the  $\Gamma$ -semigroup by weakening slightly the defining conditions of  $\Gamma$ -semigroup to ensure it preserves semigroup structure. The development of  $\Gamma$ -semigroups hinges on the fact that subsets of a semigroup naturally inherits associativity but not necessarily closed. As a result of this, various generalizations and analogues of corresponding results in semigroup theory have been obtained based on the modified definition (see [10, 15, 16, 17]).

In an attempt to broaden the theoretical aspect of  $\Gamma$ -semigroup theory, Dutta and Adhikari [4] slightly changed the defining conditions of  $\Gamma$ -semigroup by Sen and Saha [14] and then introduced the notion of left operator semigroup and right operator semigroup of a  $\Gamma$ -semigroup.

In [4], the authors described the relationship between the set of  $\Gamma$ -ideals and operator semigroups. In relation to the concept, Dutta and Chattopadhyay [5] initiated the notions of uniformly strongly prime semigroup, uniformly strongly prime ideal, Rees congruence on  $\Gamma$ -semigroup and uniformly strongly prime  $\Gamma$ -semigroup and studied these via operator semigroups of  $\Gamma$ -semigroup. Sardar *et al.* [12] showed that the left operator and right

operator semigroups of a  $\Gamma$ -semigroup with unities are Morita equivalence monoid and further established that there is a close connection between the Morita equivalence of monoids and  $\Gamma$ -semigroups. Although there is significantly number of published results in literature on operator semigroups of a  $\Gamma$ -semigroup, however the aspect of operator semigroups acting on a  $\Gamma$ -semigroup observed by [11] has not been given much attention. This serves as a motivation to write this paper and we deduce some results.

## 2 Preliminaries

We recall some definitions and results related to this paper.

**Definition 2.1** *Let  $S$  and  $\Gamma$  be two non-empty sets. Then  $S$  is called a  $\Gamma$ -semigroup if there exist mappings  $S \times \Gamma \times S \rightarrow S \mid (a, \alpha, b) \rightarrow a\alpha b \in S$  and  $\Gamma \times S \times \Gamma \rightarrow \Gamma \mid (\alpha, a, \beta) \rightarrow \alpha a \beta \in \Gamma$  satisfying the identities  $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$  for all  $a, b, c \in S$  and  $\alpha, \beta \in \Gamma$ .*

The modified definition of  $\Gamma$ -semigroup by Sen and Saha [14] may be regarded as one-sided  $\Gamma$ -semigroup.

**Definition 2.2** *Let  $S = \{a, b, c, \dots\}$  and  $\Gamma = \{\alpha, \beta, \gamma, \dots\}$  be two non-empty sets. Then  $S$  is called a  $\Gamma$ -semigroup if there exists a mapping  $S \times \Gamma \times S \rightarrow S \mid (a, \alpha, b) \rightarrow a\alpha b \in S$  satisfying the property  $(a\alpha b)\beta c = a\alpha(b\beta c)$  for all  $a, b, c \in S$  and  $\alpha, \beta \in \Gamma$ .*

A  $\Gamma$ -semigroup  $S$  is called commutative if  $x\alpha y = y\alpha x$  for every  $x, y \in S$  and  $\alpha \in \Gamma$ .

Let  $A$  and  $B$  be two subsets of a  $\Gamma$ -semigroup  $S$ . We define the set

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write  $a\Gamma B$ ,  $A\Gamma b$  and  $A\gamma B$  instead of  $\{a\}\Gamma B$ ,  $A\Gamma\{b\}$  and  $A\{\gamma\}B$  respectively.

Let  $S$  be an arbitrary semigroup and  $\Gamma$  be a non-empty set. Define a mapping  $S \times \Gamma \times S \rightarrow S$  by  $a\alpha b = ab$  for all  $a, b \in S$  and  $\alpha \in \Gamma$ . It is easy to see that  $S$  is a  $\Gamma$ -semigroup. Thus, a semigroup can be considered as a  $\Gamma$ -semigroup.

In the following, some examples of  $\Gamma$ -semigroups are presented.

**Example 2.1** Let  $S = \mathbb{Z}$  be the set of all integers and  $\Gamma = \{n \mid n \in \mathbb{N}\}$ . Then  $S$  is a  $\Gamma$ -semigroup with the operation defined by  $a\alpha b = a + \alpha + b$  for all  $a, b \in S$  and  $\alpha \in \Gamma$ .

**Example 2.2** Let  $S$  be a set of all negative rational numbers. Obviously  $S$  is not a semigroup under usual product of rational numbers. Let  $\Gamma = \{-\frac{1}{p} \mid p \text{ is prime}\}$ . Let  $a, b, c \in S$  and  $\alpha, \beta \in \Gamma$ . Now if  $a\alpha b$  is equal to the usual product of rational numbers  $a, \alpha, b$ ; then  $a\alpha b \in S$  and  $(a\alpha b)\beta c = a\alpha(b\beta c)$ . Hence  $S$  is a  $\Gamma$ -semigroup.

**Example 2.3** Let  $S = \{-i, i, 0\}$  and  $\Gamma = S$ . Then  $S$  is a  $\Gamma$ -semigroup with respect to multiplication of complex numbers whereas  $S$  does not reduce to semigroup with respect to multiplication of complex numbers.

The following definitions and theorems can be found in [5] except Definitions 2.3.

**Definition 2.3** Let  $S$  be a  $\Gamma$ -semigroup. A nonempty subset  $A$  of  $S$  is called left (right)  $\Gamma$ -ideal of  $S$  if  $S\Gamma A \subseteq A$  ( $A\Gamma S \subseteq A$ ). Further, a non-empty  $A$  of a  $\Gamma$ -semigroup  $S$  is called  $\Gamma$ -ideal if  $A$  is both a left and a right  $\Gamma$ -ideal of  $S$ .

**Definition 2.4** Let  $S$  be a  $\Gamma$ -semigroup. Let  $L$  and  $R$  be the left and right operator semigroups of the  $\Gamma$ -semigroup  $S$ . If there exist an element  $[e, \delta] \in L$  ( $[\delta, e] \in R$ ) such that  $e\delta x = x$  ( $x\delta e = x$ ) for all  $x \in S$ , then  $S$  is said to have the left unity  $[e, \delta]$  (right unity  $[\delta, e]$ ).

**Definition 2.5** Let  $S$  be a  $\Gamma$ -semigroup. We define a relation  $\rho$  on  $S \times \Gamma$  as follows:

$$(x, \alpha)\rho(y, \beta) \iff x\alpha s = y\beta s, \forall s \in S.$$

Obviously  $\rho$  is an equivalence relation. Let  $[x, \alpha]$  denote the equivalence class containing  $(x, \alpha)$ . Let  $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$ . Then  $L$  is a semigroup with respect to multiplication defined by  $[x, \alpha][y, \beta] = [x\alpha y, \beta]$ . The semigroup  $L$  is called the left operator semigroup of  $S$ . Similarly, the right operator semigroup  $R$  of a  $\Gamma$ -semigroup  $S$  is defined as  $R = \{[\alpha, x] : \alpha \in \Gamma, x \in S\}$ , where  $[\alpha, x][\beta, y] = [\alpha, x\beta y]$ , for all  $x, y \in S$  and  $\alpha, \beta \in \Gamma$ .



**Example 2.4** Using Example 2.3,

$S \times \Gamma = \{(-i, -i), (0, 0), (i, i), (-i, 0), (-i, i), (0, -i), (0, i), (i, -i), (i, 0)\}$   
and let  $\rho = \{(-i, -i), (0, 0), (i, i), (-i, i), (i, -i)\}$  be a relation on  $S \times \Gamma$ .

By routine calculation, it is obvious that  $\rho$  is an equivalence relation. The equivalence class  $[x, \alpha] = \{(y, \beta) \in \rho \mid (y, \beta)\rho(x, \alpha)\}$ . Therefore,

$$\begin{aligned} [-i, -i] &= \{(-i, -i), (i, i)\} \\ [0, 0] &= \{(0, 0)\} \\ [i, i] &= \{(i, i), (-i, -i)\} \\ [-i, i] &= \{(-i, i), (i, -i)\} \\ [i, -i] &= \{(i, -i), (-i, i)\} \end{aligned}$$

$\implies [-i, -i] = [i, i]$  and  $[-i, i] = [i, -i]$  and the set form  $L = \{[-i, -i], [i, i], [-i, i], [i, -i]\}$  is a left operator semigroup of  $S$ . Similarly, the right operator semigroup  $R$  of  $S$  can be obtained.

**Theorem 2.1** Let  $S$  be a  $\Gamma$ -semigroup. If  $[e, \delta]$  is left unity of  $S$ , then  $[e, \delta]$  is the identity element of  $L$ .

**Theorem 2.2** Let  $S$  be a  $\Gamma$ -semigroup. If  $[\mu, f]$  is right unity of  $S$ , then  $[\mu, f]$  is the identity element of  $R$ .

**Theorem 2.3** Let  $S$  be a  $\Gamma$ -semigroup with left and right unities and  $L$  be its left operator semigroup.

(i) If  $Q$  is a  $\Gamma$ -ideal of  $S$ , then  $Q^{+'}$  is a  $\Gamma$ -ideal of  $L$ .

(ii) If  $P$  is a  $\Gamma$ -ideal of  $L$ , then  $P^+$  is a  $\Gamma$ -ideal of  $S$ .

**Theorem 2.4** Let  $S$  be a  $\Gamma$ -semigroup with left and right unities and  $R$  be its right operator semigroup.

(i) If  $Q$  is a  $\Gamma$ -ideal of  $S$ , then  $Q^{+'}$  is a  $\Gamma$ -ideal of  $R$ .

(ii) If  $P$  is a  $\Gamma$ -ideal of  $R$ , then  $P^*$  is a  $\Gamma$ -ideal of  $S$ .

**Theorem 2.5** Let  $S$  be a  $\Gamma$ -semigroup with left and right unities and let  $L$  and  $R$  be its left operator semigroup and right operator semigroup respectively. Then there is an inclusion preserving bijection between the set of all  $\Gamma$ -ideals of  $S$  and the set of all  $\Gamma$ -ideals of  $L(R)$  via the mapping  $Q \longrightarrow Q^{+'}$  ( $Q \longrightarrow Q^*$ ) where  $Q$  is a  $\Gamma$ -ideal of  $S$ .

### 3 Main Results

Throughout  $S$  stands for one-sided  $\Gamma$ -semigroup unless otherwise mentioned.

The following proposition and theorem show the commutativity and isomorphism of operator semigroups.

**Proposition 3.1** *Let  $S$  be a commutative  $\Gamma$ -semigroup and  $\alpha \in \Gamma$ . Then the left operator semigroup  $L$  and the right operator semigroup  $R$  of  $S$  are commutative.*

**Proof**

The proof is straightforward.

**Theorem 3.1** *If  $S$  is a commutative  $\Gamma$ -semigroup and  $\alpha \in \Gamma$ , then the operator semigroup  $L$  and the right operator semigroup  $R$  of  $S$  are isomorphic.*

**Proof**

Define  $f : L \longrightarrow R$  by  $f([a, \alpha]) = [\alpha, a]$ . Let  $[a, \alpha] = [b, \alpha]$ . Then  $a\alpha s = b\alpha s$  for all  $s \in S$ . Since  $S$  is commutative,  $s\alpha a = s\alpha b$  for all  $s \in S$ . So,  $[\alpha, a] = [\alpha, b]$ . Thus,  $f$  is well-defined. The mapping is injective, since

$$\begin{aligned} [\alpha, a] = [\alpha, b] &\implies s\alpha a = s\alpha b \quad \forall s \in S \\ &\implies a\alpha s = b\alpha s \\ &\implies [a, \alpha] = [b, \alpha] \end{aligned}$$

Again,  $f$  is surjective, since for any  $[\alpha, a] \in R$ ,  $a \in S$  and  $\alpha \in \Gamma$ , we have  $[\alpha, a] = f([a, \alpha])$ .

Also, for all  $a, b \in S$  and  $\alpha \in \Gamma$ ,

$$\begin{aligned} f([a, \alpha][b, \alpha]) &= f([a\alpha b, \alpha]) = [\alpha, a\alpha b] = [\alpha, b\alpha a] \\ &= [\alpha, b][\alpha, a] \\ &= [\alpha, a][\alpha, b] \\ &= f([a, \alpha])f([b, \alpha]) \end{aligned}$$

So,  $f$  is a homomorphism. Therefore,  $L$  and  $R$  are isomorphic.

In the following we consider operator semigroups acting on a  $\Gamma$ -semigroup.

**Definition 3.1** Let  $S$  be a  $\Gamma$ -semigroup and its left and right operator semigroup are respectively  $L$  and  $R$ . Then for every  $a \in S$  and  $\alpha \in \Gamma$ , we define  $L \times S \longrightarrow S$  and  $S \times R \longrightarrow S$  respectively as follow:

$$[a, \alpha]s := a\alpha s \text{ and } s[\alpha, a] := s\alpha a.$$

The following remark follows from Definition 3.1.

**Remark 3.1** If  $S$  is a commutative  $\Gamma$ -semigroup, then  $L \times S = S \times R$ .

**Example 3.1** Clearly, Example 2.3 shows that  $S$  is commutative and from Example 2.4 it is easy to verify that  $L \times S = S \times R$  for every  $s \in S$  and  $\alpha \in \Gamma$ .

**Proposition 3.2** If  $A$  is an ideal of  $S$ , then  $L \times A$  is an ideal of  $L \times S$ .

**Proof**

Suppose that  $A$  is an ideal of  $S$  and  $[a, \alpha]t \in L \times A$ . Then for every  $s, t \in S$  and  $\beta \in \Gamma$ , we have  $a\alpha(s\beta t) \in A$ . Thus,  $[a, \alpha][s, \beta]t = [a\alpha s, \beta]t \in L \times A$ . Similarly, we can prove that  $[s, \beta][a, \alpha]t = [s\beta a, \alpha]t \in L \times A$ . Hence,  $L \times A$  is an ideal of  $L \times S$ .

**Theorem 3.2** There exists an inclusion preserving bijection between the set of all right ideals of  $S$  and the set all right ideals of  $L \times S$ .

**Proof**

Suppose that  $\mathcal{I}(S)$  and  $\mathcal{I}(L \times S)$  are the sets of all right ideals of  $S$  and  $L \times S$  respectively. Clearly, the mapping  $f : \mathcal{I}(S) \longrightarrow \mathcal{I}(L \times S)$  by  $f(I) = L \times I$  is well-defined. Since  $I$  is a right ideal of  $S$ ,  $I\Gamma S \subseteq I$ . Thus,  $I \subseteq L \times I$ . On the other hand, since  $S$  has a left unity,  $L \times I \subseteq (L \times I)\Gamma S \subseteq I$ . Thus,  $L \times I = I$ . Hence,  $f$  is bijective. It remains to show that  $f$  is an inclusion preserving mapping. Let  $I$  and  $J$  be two right ideals of  $S$  such that  $I \subseteq J$ . We have to show that  $L \times I \subseteq L \times J$ . Let  $[a, \alpha] \in L$ . Then for every  $s \in S$  we have  $s \in I$  and so  $[a, \alpha]s \in L \times I$ . Since  $I \subseteq J$ , we have  $s \in J$ . Hence,  $[a, \alpha]s \in L \times J$ . Therefore,  $L \times I \subseteq L \times J$ . Hence the proof.

**Remark 3.2** Similar characterisations can be proved for right operator semigroup acting on a  $\Gamma$ -semigroup and right  $\Gamma$ -ideal.

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